

Effective separability of typical entangled many-body states

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We consider two systems of harmonically trapped particles in a typical pure state of the Hilbert space defined by given values of the particle numbers and energies of the two gases. Such a state is entangled but we show that, for large systems, the resulting correlations between the two gases are identical to those of a separable mixture. This result can be generalized to other physical systems. We discuss the relation of this effective separability to the well-known existence of quantum correlations in any entangled state. We study in detail a small bipartite system and find that its correlations are well explained by the large systems results.

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According to the postulate of equal a priori probabilities, an isolated quantum system in equilibrium must be described by a microcanonical mixed state, i.e., a density matrix which commutes with the system Hamiltonian and whose only nonzero diagonal elements correspond to eigenenergies in a given energy interval. Recently, it has been shown that, for many physically relevant properties, a pure state description is also possible. An equilibrium state is characterized by its energy and possibly other thermodynamic variables such as particle number or volume. Almost all pure states in the Hilbert space defined by these parameters lead to expectation values of interest identical to those of the corresponding microcanonical mixture, provided the considered system is large enough [1, 2, 3]. Most of the effort has been devoted to deriving the canonical ensemble for a system S weakly coupled to a large heat bath B from pure states of the composite system consisting of S and B [4, 5, 6, 7]. But the pure state description of equilibrium is not restricted to the degrees of freedom of a subsystem of a larger system. For example, the density profile of a gas in a pure state of macroscopically well-defined energy is indistinguishable from that of the corresponding microcanonical state [3].

The effective equivalence of pure states with microcanonical states raises an interesting question in the context of multipartite systems. Consider two systems A and B in a typical state of a Hilbert space characterized by some parameters such as the energies of A and B . If the above discussed equivalence applies, this entangled state cannot be distinguished from the corresponding microcanonical state. But this microcanonical state is separable, i.e., is a mixture of product states, showing possibly only classical correlations between A and B . On the other hand, it has been proved that the entangled character of any pure state can be revealed by local measurements on systems A and B , irrespective of their nature and size [8, 9]. The only pure states which do not violate Bell's inequalities [10, 11] are product states and hence the considered entangled state of A and B must exhibit quantum correlations. This seems to be in contradiction

with a possible equivalence to a separable state.

In this Letter, we consider two harmonically trapped gases of bosons or fermions, in an entangled state characterized by the energies and particle numbers of the two systems. We first show that, for large systems, almost all pure states determined by these thermodynamic parameters, lead to the same bipartite correlations. These correlations are found to be identical to those of a separable mixture. For small particle numbers and energies, the model we study is simple enough to allow the complete determination of the basis spanning the corresponding Hilbert space. It is thus possible to evaluate correlations between systems A and B for particular entangled states drawn from this space. We find that these correlations are well described by the expressions obtained for large systems, even for as few as ten particles. We discuss in detail the relation of our result to the well-known violation of Bell's inequalities mentioned above.

The system A we consider consists of particles confined in a harmonic trap and is described by the Hamiltonian

$$H_A = \omega_A \sum_{k \geq 0} \left(k + \frac{1}{2} \right) c_{Ak}^\dagger c_{Ak} \quad (1)$$

where ω_A is the frequency of the harmonic confining potential and c_{Ak}^\dagger creates a particle in the single-particle eigenstate $k \in \mathbb{N}$. Throughout this paper, we use units in which $\hbar = 1$. The cases of bosons and fermions will be treated simultaneously in the following. We assume that the system A contains a number of particles N_A and has an energy E_A . The corresponding eigenstates $|\{n_{Ak}\}\rangle$ of H_A satisfy

$$\sum_{k \geq 0} n_{Ak} = N_A, \quad \sum_{k \geq 0} n_{Ak} k = M_A = \frac{E_A}{\omega_A} - \frac{N_A}{2}, \quad (2)$$

where M_A is an integer, $n_{Ak} \in \mathbb{N}$ for bosons and $n_{Ak} \in \{0, 1\}$ for fermions. The Hilbert space spanned by these states is denoted by \mathcal{H}_A and its dimension by D_A . The system B is described by a Hamiltonian of the form (1) and is characterized by a particle number N_B and an

energy $E_B = M_B + N_B/2$. We denote the corresponding eigenstates and Hilbert space by $|\{n_{Bk}\}\rangle$ and \mathcal{H}_B .

For some practical purposes, it is useful to rewrite the conditions (2) as

$$M_A = \sum_{i=1}^{N_A} k_i \quad (3)$$

where the positive integers k_i obey $k_{i+1} \geq k_i$ for bosons and $k_{i+1} > k_i$ for fermions. It is clear from this form that M_A can be as small as zero for bosons but $M_A \geq N_A(N_A - 1)/2$ for fermions. Another interesting conclusion can be drawn from (3) as follows. A fermionic configuration $\{k_i\}$ satisfying (3) with the numbers N_A and M_A corresponds to a bosonic configuration $\{k'_i = k_i - i + 1\}$ satisfying (3) with the numbers $N'_A = N_A$ and $M'_A = M_A - N_A(N_A - 1)/2$. Consequently, the Hilbert space dimension D_A is the same for a fermionic system characterized by the numbers N_A and M_A and for a bosonic system characterized by N_A and M'_A . This dimension increases with N_A and M_A . It is equal to 1 for $N_A = 1$. For $N_A = 2$, it can be easily shown from (3) that, for bosons, $D_A = M_A/2 + 1$ for even M_A and $D_A = M_A/2 + 1/2$ for odd M_A . The dimension in the fermionic case can be obtained using the fermion-boson correspondence just discussed. For larger N_A , D_A increases much faster with M_A , see Fig. 2. The results shown in this figure are obtained for $N_A = 5$ and are well approximated by $\ln(D_A) \sim \sqrt{M_A}$. For $M_A = 30$, we find $D_A = 674$. A complete discussion of D_A for large N_A and M_A can be found in Ref. [12].

We consider that the bipartite system consisting of A and B is in a pure entangled state

$$|\Psi\rangle = \sum_{\{n_{Ak}, n_{Bk}\}} \Psi_{\{n_{Ak}, n_{Bk}\}} |\{n_{Ak}\}\rangle \otimes |\{n_{Bk}\}\rangle \quad (4)$$

which belongs to the product space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $D = D_A D_B$. We are interested in the expectation values

$$\langle O_A O_B \rangle = \langle \Psi | O_A O_B | \Psi \rangle \quad (5)$$

where O_X ($X = A$ or B) is an observable of X with eigenvalues between -1 and 1 in a H_X -invariant space \mathcal{H}'_X containing \mathcal{H}_X [13]. An example is the number of particles of system X between two given positions divided by the total number N_X [3]. Bell's inequalities are written in terms of such observables [8, 9, 10, 11, 14]. We now show that, in the limit of large D , the expectation value (5) is the same for almost all states $|\Psi\rangle \in \mathcal{H}$. To do so, we use the normalized uniform measure on the unit sphere in \mathcal{H}

$$\mu(\{\Psi_{\{n_\alpha\}}\}) = \frac{(D-1)!}{\pi^D} \delta\left(1 - \sum_{\{n_\alpha\}} |\Psi_{\{n_\alpha\}}|^2\right) \quad (6)$$

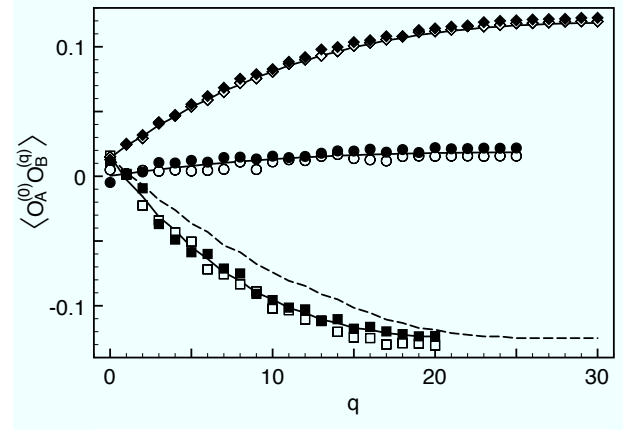


FIG. 1: Correlation $\langle O_A^{(0)} O_B^{(q)} \rangle$ as a function of q for bosons, $N_A = N_B = 5$ and $M = 20$ (squares), 25 (circles) and 30 (diamonds). For each M , two typical states (4) are shown. The lines correspond to the Hilbert space average (7). The dashed line is this average for fermions, $N_A = N_B = 5$ and $M = 30$. This curve is flat for $q \geq 25$ as $n_k = 0$ for $k \geq 25$ in this case. In general, the fermionic curve for a given M is close to the bosonic curve for $M' = M - 10$ for $0 \leq q \leq M'$, and is flat for $M - 5 \leq q \leq M$.

where $\{n_\alpha\}$ stands for $\{n_{Ak}, n_{Bk}\}$. Using the expression $\pi^p R^{2p}/p!$ for the volume of a $2p$ -dimensional sphere of radius R , we find the Hilbert space average

$$\overline{\langle O_A O_B \rangle} = \int d^2D \Psi \mu \langle O_A O_B \rangle = \langle O_A \rangle_{E_A} \langle O_B \rangle_{E_B} \quad (7)$$

where $d^2D \Psi = \prod_{\{n_\alpha\}} d\text{Re}\Psi_{\{n_\alpha\}} d\text{Im}\Psi_{\{n_\alpha\}}$,

$$\langle O_A \rangle_{E_A} = \frac{1}{D_A} \sum_{\{n_{Ak}\}} \langle \{n_{Ak}\} | O_A | \{n_{Ak}\} \rangle \quad (8)$$

and $\langle O_B \rangle_{E_B}$ is given by a similar expression. For the Hilbert space variance $\sigma^2 = \overline{\langle O_A O_B \rangle^2} - \overline{\langle O_A O_B \rangle}^2$, we obtain, by an analogous calculation,

$$\sigma^2 = \frac{1}{D^2 + D} \sum_{\{n_\alpha\}} \sum_{\{n'_\alpha\}} |\langle \{n_\alpha\} | O_A O_B | \{n'_\alpha\} \rangle|^2 - \frac{1}{D+1} \langle O_A \rangle_{E_A}^2 \langle O_B \rangle_{E_B}^2. \quad (9)$$

The above sums run only over the configurations satisfying (2). An upperbound to the variance σ^2 is thus obtained by replacing one of these sums by a sum over the set of states $|\{n_\alpha\}\rangle$ spanning $\mathcal{H}'_A \otimes \mathcal{H}'_B$. Doing so, we find $\sigma^2 < D^{-1} \langle O_A^2 \rangle_{E_A} \langle O_B^2 \rangle_{E_B} < D^{-1}$. In conclusion, in the large D limit, $\langle O_A O_B \rangle$ is given by the product $\langle O_A \rangle_{E_A} \langle O_B \rangle_{E_B}$ for almost all states $|\Psi\rangle \in \mathcal{H}$. In other words, for given observables O_A and O_B , the correlation $\langle O_A O_B \rangle$ for a typical entangled state (4) is identical to that for the separable state $D^{-1} \sum_{\{n_\alpha\}} |\{n_\alpha\}\rangle \langle \{n_\alpha\}|$.

This result seems to be in contradiction with the fact that the entangled nature of the state $|\Psi\rangle$ can be revealed by performing local measurements on A and B .

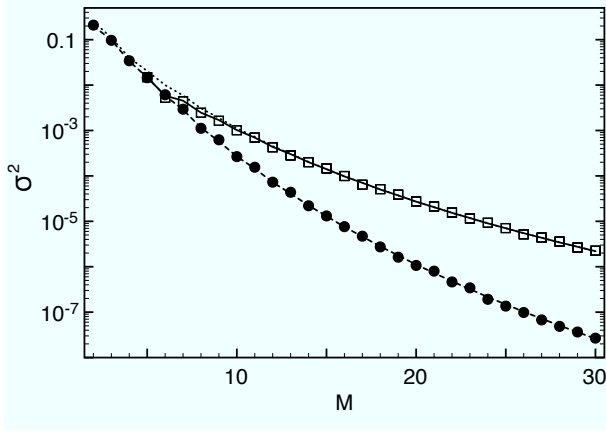


FIG. 2: Variance σ^2 as a function of M for bosons, $N_A = N_B = 5$, $(O_A, O_B) = (O_A^{(0)}, O_B^{(0)})$ (solid line and open squares) and $(O_A^{(M)}, O_B^{(M)})$ (dashed line and filled circles). We remark that σ^2 vanishes for $M \leq 4$ in the first case and for $M \leq 1$ in the second case. The lines correspond to the Hilbert space variance (9). The symbols are obtained by averaging $(\langle O_A O_B \rangle - \langle O_A \rangle_{E_A} \langle O_B \rangle_{E_B})^2$ over 10^3 typical states (4). For $M > 10$, the variance for $(O_A, O_B) = (O_A^{(p)}, O_B^{(q)})$ where $0 \leq p, q \leq M$ lies between the two curves shown in the figure. The dotted line is $D^{-1} = D_A^{-2}$.

It has been shown that, for any given entangled state of an arbitrary bipartite system (A, B) , there exist four observables O_A, O'_A, O_B and O'_B with eigenvalues ± 1 such that $|\langle F \rangle| > 2$ where $F = O_A(O_B + O'_B) + O'_A(O_B - O'_B)$ [8, 9]. On the other hand, the above argumentation can be applied to F and one finds, in the large D limit, $\langle F \rangle = \langle O_A \rangle_{E_A} \langle O_B + O'_B \rangle_{E_B} + \langle O'_A \rangle_{E_A} \langle O_B - O'_B \rangle_{E_B}$ which is clearly between -2 and 2 . In short, for any given observables O_A, O'_A, O_B and O'_B , $|\langle F \rangle| \leq 2$ for almost all entangled states (4) but, for each of these states, there exist purpose-built observables for which this inequality is violated. For large many-body systems, these very special observables might be difficult to measure as the number of measurements that can be made in practice is limited. For example, measurements of single-particle observables $\sum_{k,k'} \lambda_{kk'} c_{Xk}^\dagger c_{Xk'}$ such as particle number densities, can be achieved, whereas n -particle observables with n of the order of N_X , are practically inaccessible. Another point is worth mentioning here. In the following, we consider an observable F such that $\overline{\langle F \rangle} \rightarrow 2^-$ in the large D limit. In such a case, although $\overline{\langle F \rangle^2} - \overline{\langle F \rangle}^2 \rightarrow 0$, the proportion of states (4) such that $\langle F \rangle > 2$, does not necessarily vanish in this limit and the above conclusion may be not strictly correct. However, the proportion of states $|\Psi\rangle$ such that $\langle F \rangle > 2 + \epsilon$ where ϵ is a positive number, as small as we please, obviously vanishes.

The effective equivalence of a typical state (4) with a separable state can be generalized to other systems. First of all, for two systems A and B described by the Hamiltonian (1), other Hilbert spaces \mathcal{H}_X can be considered.

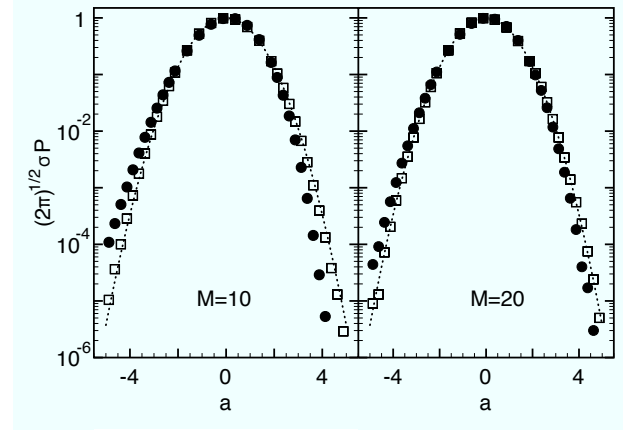


FIG. 3: Distribution $P(\langle O_A O_B \rangle)$ as a function of $a = (\langle O_A O_B \rangle - \langle O_A \rangle_{E_A} \langle O_B \rangle_{E_B}) / \sigma$ for bosons, $N_A = N_B = 5$, $(O_A, O_B) = (O_A^{(0)}, O_B^{(0)})$ (open squares) and $(O_A^{(M)}, O_B^{(M)})$ (filled circles), and $M = 10$ and 20 . The dotted lines correspond to the normal distribution of variance σ^2 .

The above derivation remains valid, for example, if the condition (3) is replaced by $\sum_{i=1}^{N_A} k_i < M_A$ or, in other words, for states $|\Psi\rangle$ given by (4) with the sum running over all the eigenstates $|\{n_{Ak}\}\rangle$ and $|\{n_{Bk}\}\rangle$ corresponding to eigenenergies lower than E_A and E_B , respectively. For other systems, such states show effective separability in a proper thermodynamic limit. It can be seen as follows. In this limit, the average (8) and $S_A = k_B \ln(D_A)$ where k_B is the Boltzmann constant, are, respectively, the usual microcanonical average and entropy [15]. As is well known, the entropy S_A is extensive and hence the variance (9) vanishes exponentially with the system size.

We now study in detail a small bipartite system. We consider two systems described by the Hamiltonian (1) with $N_A = N_B = 5$ and $M_A = M_B = M$ between 0 and 30 for bosons and between $N_A(N_A - 1)/2 = 10$ and 40 for fermions. The corresponding eigenstates are determined using (3). As observables O_X , we use the operators defined by $O_X^{(q)} |\{n_{Xk}\}\rangle = \pm |\{n_{Xk}\}\rangle$ with the upper sign if $n_{Xq} = 0$ where q is a given positive integer, and the lower if $n_{Xq} \neq 0$. For these observables, the first term of (9) simplifies to $1/(D + 1)$. We remark that $O_X^{(q)} = 1 - 2c_{Xq}^\dagger c_{Xq}$ for fermions. To draw a normalised state (4) from the uniform distribution (6), we generate D random complex numbers $\Phi_{\{n_\alpha\}}$ with standard normal distribution and then compute the components $\Psi_{\{n_\alpha\}} = \Phi_{\{n_\alpha\}} / \sum_{\{n_\alpha\}} |\Phi_{\{n_\alpha\}}|^2$ [6]. For each state, we evaluate correlations (5). Results obtained in this way are shown in Fig. 1. The agreement with the microcanonical average (7) is excellent. As discussed above, the dispersion of $\langle O_A O_B \rangle$ around this mean value decreases with increasing M , see Fig. 2. Distributions of correlations $\langle O_A^{(q)} O_B^{(q')} \rangle$ are shown in Fig. 3. They are constructed by evaluating such expectation values for 10^7

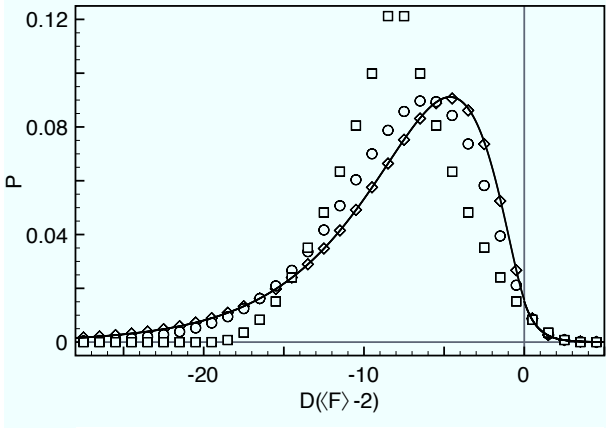


FIG. 4: Distribution $P(\langle F \rangle)$ as a function of $D(\langle F \rangle - 2)$ for bosons, $N_A = N_B = 5$ and $M = 2$ (squares), 3 (circles) and 10 (diamonds). The line is the large D expression (11).

typical states $|\Psi\rangle$. It can be shown that they converge to normal distributions as M is increased.

To discuss Bell's inequalities, let us consider two particular states $|\{n_{Ak}^\pm\}\rangle$ satisfying (2), and define observables σ_A^z and σ_A^x which act on the states $|\{n_{Ak}\}\rangle \neq |\{n_{Ak}^\pm\}\rangle$ like the identity operator, and are represented by the Pauli matrices σ_z and σ_x , respectively, in the basis $\{|\{n_{Ak}^+\}\rangle, |\{n_{Ak}^-\}\rangle\}$ of the corresponding subspace. For system B , we define the observables σ_B^u and σ_B^v which coincide with the unit operator except on the subspace spanned by two states $|\{n_{Bk}^\pm\}\rangle$ in which $\sigma_B^u = 2^{-1/2}(\sigma_B^z - \sigma_B^x)$ and $\sigma_B^v = 2^{-1/2}(\sigma_B^z + \sigma_B^x)$ where σ_B^z and σ_B^x are the analogues of σ_A^z and σ_A^x , respectively. We are interested in the expectation value

$$\langle F \rangle = \langle \sigma_A^x(\sigma_B^u + \sigma_B^v) + \sigma_A^z(\sigma_B^u - \sigma_B^v) \rangle \quad (10)$$

in a typical state (4). It can be written in terms of the four components $\Psi_{\{n_\alpha\}}$ corresponding to $|\{n_{Ak}^\pm\}\rangle, |\{n_{Bk}^\pm\}\rangle$. To simplify the following expressions, we denote these components by $\Psi_{\pm\pm}$. We first observe that, for states $|\Psi\rangle$ such that $\Psi_{++} = \Psi_{--} = 0$ and $\Psi_{+-} = \Psi_{-+}$, $\langle F \rangle = 2\sqrt{2}\eta + 2(1 - \eta)$ where $\eta = 2|\Psi_{+-}|^2 \in [0, 1]$, and hence varies between 2 and $2\sqrt{2}$ which is the maximum possible value for such an expectation value [14]. So, the CHSH inequality [11], $|\langle F \rangle| \leq 2$, is violated by some states (4). For the Hilbert space average, we find $\overline{\langle F \rangle} = 2 - 8/D < 2$. It is thus interesting to study the distribution $P(\langle F \rangle)$ resulting from the measure (6). Figure 4 shows such distributions constructed by evaluating $\langle F \rangle$ for 10^8 states $|\Psi\rangle$. We remark that P is independent of the choice of the states $|\{n_{Ak}^\pm\}\rangle$, it depends only on the dimension D . A large D expression for P can be derived as follows. For large D , the distribution of the four components $\Psi_{\pm\pm}$ is essentially Gaussian. Consequently, using $\delta(x) = \int dk \exp(ikx)/2\pi$, P can be written as the Fourier transform of a Gaussian integral which is readily

evaluated. Then a residue calculation gives

$$P \simeq \frac{D}{8} \frac{e^{-x(\sqrt{2}+1)}}{3\sqrt{2}+4} \Theta(x) + \frac{D}{8} \left(\frac{e^{x(\sqrt{2}-1)}}{3\sqrt{2}-4} + 2e^x(x-2) \right) \Theta(-x) \quad (11)$$

where $x = D(\langle F \rangle - 2)/2$. This expression agrees very well with the results obtained for M as small as 10, see Fig.4. From it, we infer that the proportion of states (4) such that $\langle F \rangle > 2$, is $(40 + 28\sqrt{2})^{-1} \simeq 0,013$.

In summary, we have studied two harmonically trapped gases in an entangled pure state characterized by the particle numbers and energies of the two systems. For almost all such states, the correlations between the two systems are identical to those of a separable mixed state. We have proved this for large systems by evaluating the Hilbert space average and variance of such a correlation, and have shown how this proof can be applied to other physical systems. We have also studied in detail a small bipartite system and found that its correlations are well explained by the large systems results. To discuss the seeming inconsistency between the effective separability found here and the unavoidable violation of Bell's inequalities, we have considered an observable which leads to a maximal violation of the CHSH inequality [11] for some of the considered states. The proportion of states which do not satisfy this inequality remains finite in the limit of large systems but the inequality violation is less and less pronounced as the system size is increased.

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